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# Fourier Series Coefficients for Powers of the Jacobian Elliptic Functions

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**Abstract.** The Fourier series expansion coefficients for the Jacobian elliptic functions $\mathrm{sn}^m(u, k)$, $\mathrm{cn}^m(u, k)$ and $\mathrm{dn}^m(u, k)$, with $m \geqslant 1$, are studied. Two-term recurrence formulae are obtained and some of the coefficients are tabulated.

**1. Introduction.** In the last few years, the papers written on the Jacobian elliptic functions (hereafter referred to as JEFs) have been in the direction of developing the properties of the Taylor series expansion coefficients of the functions $\mathrm{sn}(x, k)$, $\mathrm{cn}(x, k)$ and $\mathrm{dn}(x, k)$ (Schett [7], [8], [9], Wrigge [11], [12], Dumont [3], Fransén [5]). Wrigge [11] studied and obtained recurrence formulae for the Taylor series coefficients of $\mathrm{sn}^m(x, k)$ when $m = 1$ and $m = 2$. Although the Fourier series expansion for the twelve JEFs have been studied and given by several authors (Abramowitz and Stegun [1], Byrd and Friedman [2], Du Val [4], Whittaker and Watson [10]), no recurrence formula has been given for the coefficients of these series corresponding to powers of the JEFs. Those for $\mathrm{cn}^2(u, k)$ and $\mathrm{dn}^2(u, k)$ can be easily obtained by using fundamental relations and the expansion of $\mathrm{sn}^2(u, k)$ which is given by Whittaker and Watson [10, Art. 22.735, Ex. 5, p. 520]. In this paper, we obtain recurrence formulae for these coefficients of $\mathrm{sn}^m(u, k)$, $\mathrm{cn}^m(u, k)$ and $\mathrm{dn}^m(u, k)$, with $m = 1, 2, 3, \ldots$, in series of the same form.

**2. Definitions and Preliminaries.** The derivation of these formulae is based on the Fourier series for $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ (see (A1), (A2), (A3) in Appendix) or the following Fourier series for $\mathrm{sn}^2(u, k), \mathrm{cn}^2(u, k)$ and $\mathrm{dn}^2(u, k)$, depending on whether the power of the JEF to be expanded is odd or even.

$$\mathrm{sn}^2(u, k) = \frac{K - E}{Kk^2} - \frac{2\pi^2}{K^2k^2} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos \frac{n\pi}{K} u, \tag{1}$$

$$\mathrm{cn}^2(u, k) = \frac{E - Kk'^2}{Kk^2} + \frac{2\pi^2}{K^2k^2} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos \frac{n\pi}{K} u, \quad \text{and} \tag{2}$$

$$\mathrm{dn}^2(u, k) = \frac{E}{K} + \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos \frac{n\pi}{K} u, \tag{3}$$

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which are the periodic functions of $u$ with period $2K$ and are valid when $|\text{Im}(u/2K)| < \text{Im}(iK'/K)$. In the above, $K$ and $E$ are the complete elliptic integrals of the first and second kind, respectively, $k$ is the modulus, $k' = (1 - k^2)^{1/2}$ the complementary modulus of the elliptic functions and the quantity $q = \exp(-\pi K'/K)$ is referred to as the nome.

From the basic properties of $\text{sn}(u, k)$, $\text{cn}(u, k)$ and $\text{dn}(u, k)$ it is easily shown that

$$\frac{d^2}{du^2}\,\text{sn}^m(u,k),\quad \frac{d^2}{du^2}\,\text{cn}^m(u,k)\quad\text{and}\quad \frac{d^2}{du^2}\,\text{dn}^m(u,k)$$

with $m \geqslant 1$, successively lead to the expressions [10, Art. 22.72, Ex. 4, p. 516]:

$$k^{m+2}\text{sn}^{m+2}(u,k) = \frac{k^m}{m(m+1)}\frac{d^2}{du^2}\,\text{sn}^m(u,k) - \frac{(m-1)k^m}{(m+1)}\,\text{sn}^{m-2}(u,k) + \frac{m(k^2+1)k^m}{(m+1)}\,\text{sn}^m(u,k), \tag{4}$$

$$k^{m+2}\text{cn}^{m+2}(u,k) = -\frac{k^m}{m(m+1)}\frac{d^2}{du^2}\,\text{cn}^m(u,k) + \frac{(m-1)k'^2k^m}{(m+1)}\,\text{cn}^{m-2}(u,k) + \frac{m(k^2-k'^2)k^m}{(m+1)}\,\text{cn}^m(u,k), \tag{5}$$

$$k^{m+2}\text{dn}^{m+2}(u,k) = -\frac{k^m}{m(m+1)}\frac{d^2}{du^2}\,\text{dn}^m(u,k) - \frac{(m-1)k'^2k^{m+2}}{(m+1)}\,\text{dn}^{m-2}(u,k) + \frac{m(1+k'^2)k^{m+2}}{(m+1)}\,\text{dn}^m(u,k), \tag{6}$$

each of which expresses even (odd) powers of one of the elliptic functions in terms of even (odd) powers of the JEF of the corresponding type.

**3. Derivation of Recurrence Formulae.** To derive the recurrence relation for the Fourier series coefficients of the JEF $\text{sn}(u, k)$ of odd (even) powers, the Fourier series for

$$k^{m+2}\text{sn}^{m+2}(u,k) \tag{7}$$

and

$$\frac{d^2}{du^2}\left[k^{m+2}\text{sn}^{m+2}(u,k)\right] \tag{8}$$

are used with $m = 2r + 1$ ($m = 2r$).

Differentiating (A1) twice and using (4) with $m = 2r + 1$, the Fourier coefficients corresponding to (7) and (8) can be generated successively with the increasing values of $r$ ($r = 0, 1, 2, \ldots$). Then, by induction, we obtain the generalization for odd powers of the JEF $\mathrm{sn}(u, k)$ as

$$(9)\quad k^{2r+3}\,\mathrm{sn}^{2r+3}(u,k) = \frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\mathscr{S}_{s,n}^{(r)}\sin\frac{(2n+1)\pi}{2K}, \qquad r = -1, 0, 1, \ldots,$$

where

$$(10)\quad \left\{\begin{array}{l} \mathscr{S}_{s,n}^{(-2)} = 0, \\ \mathscr{S}_{s,n}^{(-1)} = 1, \\ \mathscr{S}_{s,n}^{(r)} = \dfrac{1}{2(r+1)}\left[\left((2r+1)(1+k^2) - \dfrac{(2n+1)^2\pi^2}{4(2r+1)K^2}\right)\mathscr{S}_{s,n}^{(r-1)} \right. \\ \qquad\qquad \left. - 2rk^2\mathscr{S}_{s,n}^{(r-2)}\right], \\ \qquad\qquad r = 0, 1, 2, \ldots \end{array}\right\}, \qquad n = 0, 1, 2, \ldots,$$

with the subscript "$s$" denoting the function $\mathrm{sn}(u, k)$.

In the case of even powers of $\mathrm{sn}(u, k)$, this relation is deduced from the Fourier coefficients of the series (7) and (8), generated successively with $m = 2r$ ($r = 1, 2, 3, \ldots$), by making use of (4). The process initially requires the differentiation of (1) twice with respect to $x$, which leads to

$$(11)\quad k^{2r+2}\,\mathrm{sn}^{2r+2}(u,k) = \mathscr{P}_s^{(r)} + \frac{2\pi^2}{K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\mathscr{R}_{s,n}^{(r)}\cos\frac{n\pi}{K}u, \qquad r = 0, 1, 2, \ldots,$$

where

$$(12)\quad \left\{\begin{array}{l} \mathscr{P}_s^{(-1)} = 1, \\ \mathscr{P}_s^{(0)} = 1 - E/K, \\ \mathscr{P}_s^{(r)} = \dfrac{1}{(2r+1)}\left[2r(k^2+1)\mathscr{P}_s^{(r-1)} - (2r-1)k^2\mathscr{P}_s^{(r-2)}\right], \end{array}\right. \qquad r = 1, 2, 3, \ldots,$$

and

$$(13)\quad \left\{\begin{array}{l} \mathscr{R}_{s,n}^{(-1)} = 0, \\ \mathscr{R}_{s,n}^{(0)} = -1, \\ \mathscr{R}_{s,n}^{(r)} = \dfrac{1}{(2r+1)}\left[\left(2r(k^2+1) - \dfrac{n^2\pi^2}{2rK^2}\right)\mathscr{R}_{s,n}^{(r-1)} \right. \\ \qquad\qquad \left. - (2r-1)k^2\mathscr{R}_{s,n}^{(r-2)}\right], \\ \qquad\qquad r = 1, 2, \ldots \end{array}\right\}, \qquad n = 1, 2, \ldots.$$

Following a similar procedure, the two-term recurrence relations for the coefficients of odd and even powers of the remaining two types of JEFs can be found as

$$
(14)\quad k^{2r+3}\operatorname{cn}^{2r+3}(u,k) = \frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\mathcal{S}_{c,n}^{(r)}\cos\frac{(2n+1)\pi}{2K}u, \qquad r=-1,0,1,2,\ldots,
$$

with

$$
(15)\quad \left.\begin{cases}\mathcal{S}_{c,n}^{(-2)}=0,\\ \mathcal{S}_{c,n}^{(-1)}=1,\\ \mathcal{S}_{c,n}^{(r)}=\dfrac{1}{2(r+1)}\left[\left((2r+1)(k^2-k'^2)+\dfrac{(2n+1)^2\pi^2}{4(2r+1)K^2}\right)\mathcal{S}_{c,n}^{(r-1)}\right.\\ \qquad\qquad \left.+2rk^2k'^2\mathcal{S}_{c,n}^{(r-2)}\right], \\ \qquad\qquad r=0,1,2,\ldots\end{cases}\right\},
$$

$$
n=0,1,\ldots .
$$

$$
(16)\quad k^{2r+2}\operatorname{cn}^{2r+2}(u,k) = \mathcal{P}_c^{(r)}+\frac{2\pi^2}{K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\mathcal{R}_{c,n}^{(r)}\cos\frac{n\pi}{K}u, \qquad r=0,1,2,\ldots,
$$

with

$$
(17)\quad \begin{cases}\mathcal{P}_c^{(-1)}=1,\\ \mathcal{P}_c^{(0)}=E/K-k'^2,\\ \mathcal{P}_c^{(r)}=\dfrac{1}{(2r+1)}\left[2r(k^2-k'^2)\mathcal{P}_c^{(r-1)}+(2r-1)k^2k'^2\mathcal{P}_c^{(r-2)}\right],\end{cases}
$$

$$
r=1,2,3,\ldots,
$$

and

$$
(18)\quad \left.\begin{cases}\mathcal{R}_{c,n}^{(-1)}=0,\\ \mathcal{R}_{c,n}^{(0)}=1,\\ \mathcal{R}_{c,n}^{(r)}=\dfrac{1}{(2r+1)}\left[\left(2r(k^2-k'^2)+\dfrac{n^2\pi^2}{2rK^2}\right)\mathcal{R}_{c,n}^{(r-1)}\right.\\ \qquad\qquad \left.+(2r-1)k^2k'^2\mathcal{R}_{c,n}^{(r-2)}\right], \\ \qquad\qquad r=1,2,\ldots\end{cases}\right\}, \qquad n=1,2,\ldots .
$$

$$(19)\quad k^{2r+3}\,\mathrm{dn}^{2r+3}(u,k) = \mathscr{Q}_d^{(r)} + \frac{2k\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n+1}}{1+q^{2(n+1)}}\mathscr{S}_{d,n}^{(r)}\cos\frac{(n+1)\pi}{K}u,\qquad r=-1,0,1,\ldots,$$

with

$$(20)\quad \begin{cases} \mathscr{Q}_d^{(-2)} = 0,\\ \mathscr{Q}_d^{(-1)} = k\pi/2K,\\ \mathscr{Q}_d^{(r)} = \dfrac{k^2}{2(r+1)}\left[(2r+1)(1+k'^2)\mathscr{Q}_d^{(r-1)} - 2rk^2k'^2\mathscr{Q}_d^{(r-2)}\right], \end{cases}$$

$$r=0,1,2,\ldots,$$

and

$$(21)\quad \left.\begin{cases} \mathscr{S}_{d,n}^{(-2)} = 0,\\ \mathscr{S}_{d,n}^{(-1)} = 1,\\ \mathscr{S}_{d,n}^{(r)} = \dfrac{k^2}{2(r+1)}\left[\left((2r+1)(1+k'^2) + \dfrac{(n+1)^2\pi^2}{(2r+1)K^2}\right)\mathscr{S}_{d,n}^{(r-1)} - 2rk^2k'^2\mathscr{S}_{d,n}^{(r-2)}\right],\\ \qquad r=0,1,2,\ldots \end{cases}\right\},$$

$$n=0,1,2,\ldots.$$

$$(22)\quad k^{2r+2}\,\mathrm{dn}^{2r+2}(u,k) = \mathscr{P}_d^{(r)} + \frac{2\pi^2k^2}{K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\mathscr{R}_{d,n}^{(r)}\cos\frac{n\pi}{K}u,\qquad r=0,1,2,\ldots,$$

with

$$(23)\quad \begin{cases} \mathscr{P}_d^{(-1)} = 1,\\ \mathscr{P}_d^{(0)} = Ek^2/K,\\ \mathscr{P}_d^{(r)} = \dfrac{k^2}{(2r+1)}\left[2r(1+k'^2)\mathscr{P}_d^{(r-1)} - (2r-1)k^2k'^2\mathscr{P}_d^{(r-2)}\right], \end{cases}$$

$$r=1,2,3,\ldots,$$

and

$$(24)\quad \begin{cases} \mathscr{R}_{d,n}^{(-1)} = 0, \\ \mathscr{R}_{d,n}^{(0)} = 1, \\ \mathscr{R}_{d,n}^{(r)} = \dfrac{1}{(2r+1)}\left[\left(2r(1+k'^2) + \dfrac{n^2\pi^2}{2rK^2}\right)\mathscr{R}_{d,n}^{(r-1)} \right. \\ \qquad\qquad \left. + (2r-1)k^2k'^2\mathscr{R}_{d,n}^{(r-2)}\right], \\ \qquad r = 1,2,\ldots \end{cases}, \qquad n = 1,2,\ldots .$$

In this paper, the constant term and the expansion coefficients of the Fourier series for odd (even) powers of JEFs are denoted by $\mathscr{Q}$ ($\mathscr{P}$) and $\mathscr{S}(\mathscr{R})$, respectively.

**4. Results.** In this section we give numerical results for some of these coefficients. It is clear from (9), (10),...,(24) that the accuracy of the results depends strongly on that of $K, E$ and $q$. For this reason the latter quantities were computed to ten decimal places prior to doing the computations of the coefficients themselves. All of the computations were done at Middle East Technical University (Ankara) on the Burroughs 6900 using single-precision FORTRAN.

TABLE 1

*Values of $\mathscr{P}_s^{(r)}, \mathscr{P}_c^{(r)}, \mathscr{P}_d^{(r)}, \mathscr{Q}_d^{(r)}$ for $k^2 = 0.7$*

| | r=-1 | r=0 | r=1 | r=2 |
|---|---|---|---|---|
| $P_s^{(r)}$ | 1 | 0.40170 92494 | 0.22193 71494 | 0.13311 66385 |
| $P_c^{(r)}$ | 1 | 0.29829 07506 | 0.14954 42002 | 0.08543 87786 |
| $P_d^{(r)}$ | 1 | 0.41880 35254 | 0.20507 41387 | 0.11235 55021 |
| $Q_d^{(r)}$ | 0.63324 94173 | 0.28812 84849 | 0.15010 38588 | 0.08559 21680 |

TABLE 2

*Values of* $\frac{2\pi}{K}\,\frac{q^{n+1/2}}{1-q^{2n+1}}\mathcal{S}_{s,n}^{(r)}$ *for* $k^2 = 0.7$

| n | r = -1 | r=0 | r = 1 | r = 2 |
|---|---|---|---|---|
| 0 | 0.89418 94747 | 0.50393 65582 | 0.30549 55347 | 0.19178 13668 |
| 1 | 0.06182 44070 | -0.10682 56608 | -0.11194 38256 | -0.08949 65338 |
| 2 | 0.00461 57482 | -0.02912 90657 | -0.00399 04996 | 0.00984 53677 |
| 3 | 0.00034 47492 | -0.00454 55761 | 0.00471 67163 | 0.00438 99563 |
| 4 | 0.00002 57403 | -0.00057 55223 | 0.00148 26451 | 0.00007 57350 |
| 5 | 0.00000 19232 | -0.00006 50205 | 0.00029 20089 | -0.00023 06795 |
| 6 | 0.00000 01436 | -0.00000 68313 | 0.00004 63537 | -0.00008 07339 |
| 7 | 0.00000 00107 | -0.00000 06823 | 0.00000 64552 | -0.00001 82715 |
| 8 | 0.00000 00008 | -0.00000 00657 | 0.00000 08218 | -0.00000 33403 |
| 9 | 0 | -0.00000 00061 | 0.00000 00979 | -0.00000 05335 |
| 10 | 0 | -0.00000 00006 | 0.00000 00111 | -0.00000 00774 |
| 11 | 0 | 0 | 0 | -0.00000 00105 |
| 12 | 0 | 0 | 0 | -0.00000 00013 |

TABLE 3

Values of $\frac{2\pi^2}{K^2}\frac{nq^n}{1-q^{2n}}\mathcal{R}_{s,n}^{(r)}$ for $k^2 = 0.7$

| n | r=0 | r=1 | r=2 | r=3 |
|---|---|---|---|---|
| 1 | -0.34421 77075 | -0.25865 34388 | -0.17756 25892 | -0.11971 97943 |
| 2 | -0.05113 39471 | 0.02016 23245 | 0.03965 68027 | 0.03905 00051 |
| 3 | -0.00572 86101 | 0.01319 78635 | 0.00674 60642 | -0.00008 14475 |
| 4 | -0.00057 04927 | 0.00283 94654 | -0.00110 39285 | -0.00206 46556 |
| 5 | -0.00005 32626 | 0.00044 81727 | -0.00065 18249 | -0.00028 48240 |
| 6 | -0.00000 47738 | 0.00006 02236 | -0.00016 44904 | 0.00005 32787 |
| 7 | -0.00000 04160 | 0.00000 73131 | -0.00003 09355 | 0.00003 39680 |
| 8 | -0.00000 00355 | 0.00000 08277 | -0.00000 49284 | 0.00000 96135 |
| 9 | -0.00000 00030 | 0.00000 00889 | -0.00000 07030 | 0.00000 20379 |
| 10 | -0.00000 00002 | 0.00000 00092 | -0.00000 00925 | 0.00000 03655 |
| 11 | 0 | 0.00000 00009 | -0.00000 00115 | 0.00000 00585 |
| 12 | 0 | 0.00000 00001 | -0.00000 00014 | 0.00000 00086 |
| 13 | 0 | 0 | -0.00000 00002 | 0.00000 00012 |

TABLE 4

*Values of* $\frac{2\pi}{K}\frac{q^{n+1/2}}{1+q^{2n+1}}\mathscr{S}_{c,n}^{(r)}$ *for* $k^2 = 0.7$

| n | r=-1 | r=0 | r=1 | r=2 |
|---|---|---|---|---|
| 0 | 0.76989 88143 | 0.37450 34335 | 0.04938 99502 | -0.03502 40397 |
| 1 | 0.06177 29084 | 0.17159 82308 | 0.11872 01550 | 0.03595 27848 |
| 2 | 0.00461 57267 | 0.03397 54433 | 0.05025 65433 | 0.03598 74242 |
| 3 | 0.00034 47492 | 0.00490 75626 | 0.01291 58150 | 0.01570 32559 |
| 4 | 0.00002 57493 | 0.00060 25591 | 0.00250 80590 | 0.00463 09588 |
| 5 | 0.00000 19232 | 0.00006 70399 | 0.00040 71579 | 0.00106 70931 |
| 6 | 0.00000 01436 | 0.00000 69821 | 0.00005 84103 | 0.00020 69905 |
| 7 | 0.00000 00107 | 0.00000 06936 | 0.00000 76569 | 0.00003 53530 |
| 8 | 0.00000 00008 | 0.00000 00665 | 0.00000 09372 | 0.00000 54754 |
| 9 | 0 | 0.00000 00062 | 0.00000 01087 | 0.00000 07848 |
| 10 | 0 | 0.00000 00006 | 0.00000 00121 | 0.00000 01056 |
| 11 | 0 | 0 | 0.00000 00013 | 0.00000 00135 |
| 12 | 0 | 0 | 0 | 0.00000 00017 |

TABLE 5

*Values of* $\frac{2\pi^2}{K^2}\frac{nq^n}{1-q^{2n}}\mathcal{R}^{(r)}_{c,n}$ *for* $k^2 = 0.7$

| n | r = 0 | r = 1 | r = 2 | r = 3 |
|---|---|---|---|---|
| 1 | 0.34421 77075 | 0.22325 13517 | 0.14039 03977 | 0.08928 10400 |
| 2 | 0.05113 39471 | 0.09174 98504 | 0.07785 09810 | 0.05744 39671 |
| 3 | 0.00572 86101 | 0.02121 79177 | 0.02939 05060 | 0.02769 09444 |
| 4 | 0.00057 04927 | 0.00363 81551 | 0.00790 54300 | 0.01015 70883 |
| 5 | 0.00005 32626 | 0.00052 27403 | 0.00167 12836 | 0.00293 09898 |
| 6 | 0.00000 47738 | 0.00006 69069 | 0.00029 79775 | 0.00069 74589 |
| 7 | 0.00000 04160 | 0.00000 78954 | 0.00004 69044 | 0.00014 26585 |
| 8 | 0.00000 00355 | 0.00000 08774 | 0.00000 67187 | 0.00002 58950 |
| 9 | 0.00000 00030 | 0.00000 00931 | 0.00000 08941 | 0.00000 42718 |
| 10 | 0.00000 00002 | 0.00000 00095 | 0.00000 01122 | 0.00000 06519 |
| 11 | 0 | 0.00000 00009 | 0.00000 00134 | 0.00000 00933 |

TABLE 6

*Values of* $\frac{2k\pi}{K}\frac{q^{n+1}}{1+q^{2(n+1)}}\mathscr{S}_{d,n}^{(r)}$ *for* $k^2 = 0.7$

| n | r=-1 | r=0 | r=1 | r=2 |
|---|---|---|---|---|
| 0 | 0.55779 80123 | 0.70115 74837 | 0.53126 43400 | 0.36256 72995 |
| 1 | 0.04189 30129 | 0.15345 57231 | 0.18370 30417 | 0.16355 79089 |
| 2 | 0.00312 90836 | 0.02400 97302 | 0.04504 07061 | 0.05347 67638 |
| 3 | 0.00023 37111 | 0.00310 53557 | 0.00874 36281 | 0.01380 62322 |
| 4 | 0.00001 74559 | 0.00035 79368 | 0.00143 91288 | 0.00297 99203 |
| 5 | 0.00000 13038 | 0.00003 82363 | 0.00020 99960 | 0.00055 97050 |
| 6 | 0.00000 00974 | 0.00000 38712 | 0.00002 79900 | 0.00009 41773 |
| 7 | 0.00000 00073 | 0.00000 03766 | 0.00000 34785 | 0.00001 45042 |
| 8 | 0.00000 00005 | 0.00000 00355 | 0.00000 04090 | 0.00000 20779 |
| 9 | 0 | 0.00000 00033 | 0.00000 00460 | 0.00000 02804 |
| 10 | 0 | 0.00000 00003 | 0.00000 00050 | 0.00000 00360 |
| 11 | 0 | 0 | 0.00000 00005 | 0.00000 00044 |
| 12 | 0 | 0 | 0 | 0.00000 00005 |

TABLE 7

Values of $\frac{2\pi^2 k^2}{K^2} \frac{nq^n}{1-q^{2n}} \mathscr{R}_{d,n}^{(r)}$ for $k^2 = 0.7$

| n | r = 0 | r = 1 | r = 2 | r = 3 |
|---|---|---|---|---|
| 1 | 0.24095 23953 | 0.21059 31683 | 0.14894 96005 | 0.09975 69304 |
| 2 | 0.03579 37630 | 0.05999 08071 | 0.05976 15801 | 0.04944 44044 |
| 3 | 0.00401 00271 | 0.01208 09910 | 0.01716 14413 | 0.01801 61237 |
| 4 | 0.00039 93449 | 0.00195 04208 | 0.00388 74943 | 0.00520 29241 |
| 5 | 0.00003 72838 | 0.00027 18019 | 0.00073 95528 | 0.00125 22174 |
| 6 | 0.00000 33417 | 0.00003 41879 | 0.00012 33026 | 0.00026 21117 |
| 7 | 0.00000 02912 | 0.00000 39911 | 0.00001 85641 | 0.00004 88009 |
| 8 | 0.00000 00249 | 0.00000 04403 | 0.00000 25786 | 0.00000 82678 |
| 9 | 0.00000 00021 | 0.00000 00465 | 0.00000 03357 | 0.00000 12954 |
| 10 | 0.00000 00002 | 0.00000 00047 | 0.00000 00414 | 0.00000 01901 |
| 11 | 0 | 0.00000 00005 | 0.00000 00049 | 0.00000 00264 |
| 12 | 0 | 0 | 0.00000 00006 | 0.00000 00035 |

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## Appendix

Fourier series for the Jacobian elliptic functions [2, pp. 304, 305]:

$$\text{(A1)} \quad \operatorname{sn}(u,k) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin \frac{(2n+1)\pi}{2K} u,$$

$$\left[ |\operatorname{Im}(u/K)| < \operatorname{Im}(iK'/K) \right],$$

$$\text{(A2)} \quad \operatorname{cn}(u, k) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \cos \frac{(2n+1)\pi}{2K} u,$$

$$\left[ |\operatorname{Im}(u/K)| < \operatorname{Im}(iK'/K) \right],$$

$$\text{(A3)} \quad \operatorname{dn}(u, k) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 + q^{2(n+1)}} \cos \frac{(n+1)\pi}{K} u,$$

$$\left[ |\operatorname{Im}(u/K)| < \operatorname{Im}(iK'/K) \right],$$

where the periods for (A1) and (A2) are $4K$ and that for (A3) is $2K$.

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